

Gallai's Path Decomposition of Cayley Graph of Finite Abelian Group

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Extended abstract

A path decomposition of a graph G is a collection of paths \mathcal{P} such that each edge of G is present exactly in one path of \mathcal{P} . The number of paths in \mathcal{P} is called the size of the decomposition. The path number $pn(G)$ is the size of a path decomposition which is minimum among all possible path decompositions of the graph G . Gallai's path decomposition conjecture states that if G is a connected graph on n vertices, then $pn(G) \leq \lceil \frac{n}{2} \rceil$. One can observe that the conjecture is sharp for graphs where all vertices have odd degrees. Because the odd degree condition forces each of the vertices in the graph to be an end point of a path in any path decomposition of the graph. The conjecture is verified only for some specific family of graphs, like graphs having restrictions on degree condition ([2, 3, 7, 8, 13, 14]) and structural condition ([1, 5, 12]). The conjecture is still open for many well-known families of graphs.

Cayley graphs are one of the well known graph family frequently appears in algebraic graph theory and network design. Let Γ be a group and $S \subseteq \Gamma$ be a set of group elements such that the identity $e \notin S$. The Cayley graph $G = \text{Cay}(\Gamma, S)$ is a graph where the vertex set is Γ , and there is a directed edge from the vertex g to the vertex h if and only if $hg^{-1} \in S$. If $S = S^{-1}$, the Cayley graph G is an undirected graph; if S contains a generating set, G is a connected graph. Cayley graphs are highly symmetric and vertex transitive. For example Hypercube graph is a well known example of a Cayley graph.

In this work, we establish the validity of Gallai's conjecture for the class of Cayley graphs generated by the standard minimal generating set S of any finite abelian group (with one specific exception). Precisely, let Γ be a finite abelian group of order n . Then by the fundamental theorem of finite abelian groups, every such group Γ is isomorphic to a direct product of cyclic groups: $\Gamma \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, where $n = \prod_{i=1}^r n_i$ and \mathbb{Z}_m is the group of residue classes $\{0, 1, \dots, m-1\}$ under the operation of addition modulo m . Let $S = \{e_1, e_2, \dots, e_r\} \subseteq \Gamma$ be the standard generating set, where e_i is the vector with 1 in the i -th component and 0 elsewhere. Then the Cayley graph that we consider in this work is $G = \text{Cay}(\Gamma, S \cup S^{-1})$, where Γ is a finite abelian group without a \mathbb{Z}_3 in the cyclic decomposition.

Main results in our work:

Theorem 1. *Let $\Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ be a finite Abelian group of order $n = \prod_{i=1}^r n_i$ with $n_i \neq 3$. Let $S = \{e_1, e_2, \dots, e_r\}$ be the standard minimal generating set, where e_i is the vector with 1 in the i -th component and 0 elsewhere. Then the path number of the Cayley graph $G = \text{Cay}(\Gamma, S \cup S^{-1})$ satisfies: $pn(G) \leq \lceil \frac{n}{2} \rceil$.*

Sketch of the proof:

The proof exploits the fact that for the chosen standard generating set S , the graph $G = \text{Cay}(\Gamma, S \cup S^{-1})$ is isomorphic to the Cartesian product of Cayley graphs corresponding to the cyclic factors of Γ . Our strategy leverages the structural properties of Cartesian products to decompose the edge set into manageable components. By applying the known result that Gallai's conjecture holds for odd graphs (graphs where every vertex has an odd degree), we extend the construction to the entire product. This approach allows us to demonstrate that the path number satisfies the required bound while accounting for the parity of the vertices in each cyclic factor.

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